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What is This?
Non-stationary random vibration of a coupled vehicle–slab track system using a parallel algorithm based on the pseudo excitation method

Jian Zhang1, Yan Zhao1, Ya-hui Zhang1, Xue-song Jin2, Wan-xie Zhong1, Frederic W Williams3 and David Kennedy3

Abstract
A parallel algorithm based on the pseudo excitation method (PEM) is applied to investigate the non-stationary random response of a vertically coupled vehicle–slab track system subjected to random excitation induced by track irregularity. The vehicle is simplified as a multibody system with 10 degrees of freedom and the slab track is represented by a three layer Bernoulli–Euler beam model which includes the rail, slab and roadbed. Linear wheel–rail contact provides interaction between the vehicle and slab track models. The track irregularity is assumed to be a Gaussian random process with zero mean value and the non-stationary random response of the coupled vehicle–slab track system is solved using a combination of PEM and step-by-step integration. A parallel algorithm based on PEM is used to accelerate the computation and the effectiveness of the proposed method is verified by the Monte Carlo method. The basic random performance of the coupled vehicle–slab track system is analysed and the effects of the vehicle speed and slab track parameters on the random responses of the vehicle and slab track are investigated. The results indicate that the vehicle speed has a significant effect on the random response of the coupled vehicle–slab track system and that the rail pad parameters are the most important of the slab track parameters discussed.

Keywords
Vehicle–slab track interaction, random vibration, pseudo excitation method

Introduction
The two main types of railway track structure are ballasted track and slab track. Of these, the slab track has a lower maintenance cost, better stiffness uniformity and higher running stability1–3 but it has a higher construction cost. Slab track systems are used for high-speed lines in Europe, China and Japan.2,4 As the train speed increases the dynamic responses of the vehicle and track become more severe2 and this adversely affects vehicle and track maintenance cost, ride comfort and vehicle running safety. The vibrations of the vehicle and track are mainly induced by random track irregularity6 and it is necessary to calculate the random responses of the vehicle and track in order to understand their essential vibration characteristics.

Xiang et al.7 presented a new track segment element to analyse three-dimensional coupled vehicle–slab track dynamics. The vertical displacement of the slab was calculated simply by the lateral finite strip method, and the responses of the track displacement and wheel–rail force were investigated. Lei and Zhang2 proposed a new type of slab track element to calculate the vibration of a vertical coupled vehicle–slab track system. The global mass, damping and stiffness matrices were efficiently assembled by using this slab track element and this led to an investigation of the effects of various slab track parameters on the dynamic responses of the vehicle and track. The vibrations of the vehicle and track were analysed in the transition zone between the slab track and ballasted track. Multibody dynamics software was used with a

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Timoshenko flexible track model to calculate the dynamic responses of two types of innovative slab track for comparison with a ballasted track with the uneven stiffness of the slab track forming the excitation of the vehicle–track interaction. The dynamic response of a coupled vehicle–slab track system was solved using Green’s functions in the frequency and time domains, with the vehicle model simplified as a two-mass oscillator (ignoring the bogie pitch) and the slab track was treated as an infinite two-layer beam. A polygonal wheel and rail corrugation were considered to simulate the interaction between vehicle and slab track. Bitzenbauer and Dinkel presented a semi-analytical method based on Fourier transforms to calculate the dynamic response of a coupled vehicle–slab track system subjected to cosine irregularity, by representing the slab track as a continuously supported plate and the rail and slab simulated by a Bernoulli–Euler beam and with irregularity at rail joints considered when calculating the dynamic responses of the vehicle and slab track. Zhai et al. then developed a three-dimensional dynamic model in which the slab was described as an elastic plane mounted on a viscoelastic foundation. Xiao et al. established a coupled vehicle–slab track model to investigate the influence of vertical and lateral earthquake amplitudes and vehicle speed on the running safety of a vehicle, with the slab model simulated by solid finite elements. Different coupled vehicle–track dynamic models were established to investigate the difference between ballasted track and slab track. The dynamic responses of the vehicle and track were solved by using NASTRAN to compute the eigenmodes of the rail and slab, which were integrated into SIMPACK, a commercial multibody system software package. A spatial sample of the random track irregularity was obtained from the power spectral density (PSD) function of the track irregularity. The track irregularity in the time domain was considered as the external excitation to calculate the responses of the wheel–rail force and rail pad force. Luo presented a vertically coupled vehicle–slab track model in which the slab track was modeled by thin shell elements and the vehicle–slab track system was excited by a spatial sample of the random track irregularity in the time domain. Hence, the dynamic responses of the vehicle and slab track were compared by changing the slab track parameters. Galvin et al. developed a three-dimensional coupled train–track–soil model to investigate the dynamic interaction in the transition zone between ballasted track and slab track. The vehicle was simplified as a multibody model in which the interaction between two axles linked to the same bogie was ignored and the track and soil were modeled by a combination of the finite element method and the boundary element method. The spatial sample of the track irregularity was generated from the PSD to form the external excitation. Galvin et al. compared the dynamic response of slab and ballasted tracks using a similar method.

Although random track irregularity plays a significant role in inducing interaction between vehicle and slab track, at present few authors have investigated the random vibration of coupled vehicle–slab track systems. Monte Carlo methods have been used in which the spatial sample of the track irregularity is generated from the PSD as the external excitation. However, many spatial samples must be generated for accurate calculation of the random performance of the coupled vehicle–ballasted track system, resulting in high computational costs. Lin and Zhang proposed the use of the pseudo excitation method (PEM) to efficiently and accurately calculate the stationary/non-stationary random vibration of large-scale structures. This method has been successfully applied to analyze the random vibration of coupled vehicle–ballasted track and vehicle–bridge systems. However, the algorithm has now been developed to remove the restrictions which meant that in Zhang et al. and Lu et al. it could only be applied to calculate the random response of the periodic track and in Zhang et al. it was not suitable for computation of random responses of large-scale structures. Although PEM is well suited for parallel computation due to the independence of calculations at each frequency point, such parallel computation has rarely been applied to analyze large-scale coupled vehicle–structure systems.

In this paper a vertically coupled vehicle–slab track model is established to analyze the non-stationary random vibration of the vehicle and slab track. The vehicle is treated as a multibody model in which the elasticity of the car body, bogies and wheelsets is ignored. The slab track is considered as flexible and the rail, slab and roadbed are modeled by Bernoulli–Euler beam theory. The vehicle and slab track are coupled through a linearized wheel–rail force and the non-stationary random vibration of the coupled vehicle–slab track system is obtained by a combination of PEM and step-by-step integration. A parallel algorithm based on PEM is adopted to improve computational efficiency. This paper extensively investigates the influence of the vehicle speed and slab track parameters on the random responses of the vehicle and slab track.

Vertically coupled vehicle–slab track dynamics model

A vertically coupled vehicle–slab track dynamics model is established as shown in Figure 1. It is assumed that the vehicle runs along the slab track at a constant velocity. The vehicle model is simplified as a multi-rigid-body system which consists of one car.
body mounted on two bogies, four wheelsets, four primary suspensions and two secondary suspensions. The primary and secondary suspensions are represented by parallel spring-damping elements. The car body and bogies are connected through the secondary suspensions. Each bogie is supported on two wheelsets, which are linked to the bogie through the primary suspensions. The car body and the two bogies each undergo vertical displacement and pitch rotation, but only vertical displacement is considered in each primary suspension. The car body and the two bogies are connected through the secondary suspensions. Each bogie is supported on two wheelsets. Thus, there are 10 degrees of freedom in the vehicle model. The bogies and wheelsets are assumed to be numbered from right to left in Figure 1.

The equation of motion for the vehicle model is

$$M \ddot{x}_v(t) + C \dot{x}_v(t) + K_x x_v(t) = f_v(x_v(t), x_v(t))$$

where $M$, $C$, and $K_x$ are the mass, damping and stiffness matrices of the vehicle model, respectively, $\ddot{x}_v(t)$, $\dot{x}_v(t)$ and $x_v(t)$ are the acceleration, velocity and displacement vectors of the vehicle model, respectively, $x_v(t)$ denotes the displacement vector of the vehicle model and $f_v(x_v(t), x_v(t))$ denotes the external load vector of the vehicle model.

The equation of motion for the slab track is

$$M_s \ddot{x}_{s1}(t) + C_s \dot{x}_{s1}(t) + K_{s1} x_{s1}(t) = f_{s1}(x_{s1}(t), x_{s1}(t))$$

where $M_s$, $C_s$ and $K_{s1}$ are the mass, damping and stiffness matrices of the slab track model, respectively, $\ddot{x}_{s1}(t)$ and $\dot{x}_{s1}(t)$ are the acceleration and velocity vectors of the slab track model, respectively and $f_{s1}(x_{s1}(t), x_{s1}(t))$ denotes the external load vector of the slab track model.

The vehicle model and the slab track model are coupled by the linearized wheel–rail force, which is expressed as

$$f_{wr} = k_h (x_w - x_r - x_{irr})$$

where $k_h$ denotes the contact stiffness of the wheel and rail, $x_w$ denotes the wheel displacement, $x_r$ denotes the rail displacement at the wheel position and $x_{irr}$ denotes the track irregularity.

The contact stiffness of the wheel and rail is expressed as

$$k_h = 1.5 \frac{1}{G} P_0^{1/3}$$

where $G$ denotes the Hertzian constant and $P_0$ denotes the static wheel load.

$G$ is expressed for a worn wheel profile as

$$G = 3.86 R^{-0.115} \times 10^{-9} \text{m/N}^{2/3}$$

where $R$ is the wheel radius.

According to equations (1), (2) and (3), the equation of motion for the coupled vehicle–slab track system can be written as

$$M_{vt} \ddot{x}_{vt}(t) + C_{vt} \dot{x}_{vt}(t) + K_{vt} x_{vt}(t) = f_{vt}(t)$$

where $M_{vt}$, $C_{vt}$ and $K_{vt}(t)$ are the mass, damping and stiffness matrices of the coupled vehicle–slab track model, respectively, $\ddot{x}_{vt}(t)$, $\dot{x}_{vt}(t)$ and $x_{vt}(t)$ are the acceleration, velocity and displacement vectors of the coupled vehicle–track model, respectively and $f_{vt}(t)$ is the external load vector which is determined by the track irregularity.

In equation (6) the stiffness matrix is time-dependent and is determined by the contact location.
between wheel and rail. The stiffness matrix is updated at each time step and the time-variant term of the stiffness matrix is transferred into the right side of the equation (6) for convenient calculation of the system response.12,19,21,22 The coupled vehicle–slab track system is divided into two time-invariant subsystems: the vehicle subsystem and the slab track subsystem. The interaction between these subsystems is accomplished through the wheel–rail force. The responses of the vehicle and slab track can be obtained independently using step-by-step integration.

**PEM for linear time-variant system of fully coherent multiple excitation**

The equation of motion for the linear time-variant systems is

\[ M(t)\ddot{x}(t) + C(t)x(t) + K(t)x(t) = f(t) \]  

\[ \text{(7)} \]

where \( M(t), C(t) \) and \( K(t) \) are the mass, damping and stiffness matrices of the linear time-variant system, respectively, \( \ddot{x}(t), x(t) \) and \( x(t) \) are the acceleration, velocity and displacement vectors of the linear time-variant system, respectively, and \( f(t) \) denotes the random external force vector.

The fully coherent multiple excitation is expressed as

\[ r(t) = [r(t - t_1), r(t - t_2), \ldots, r(t - t_n)]^T \]  

\[ \text{(8)} \]

where the superscript \( T \) denotes transpose, the random excitation \( r(t) \) is a stationary random Gaussian process with zero mean value, \( t_i \) denotes the time lag with \( t_1 = 0 \) and \( n \) is the number of excitations.

The random external force is expressed as

\[ f(t) = \Gamma(t)r(t) \]  

\[ \text{(9)} \]

where \( \Gamma(t) \) denotes the index matrix representing the external force location, whose elements are either zero or one.

Using Duhamel’s integral, an arbitrary response induced by the random external force can be expressed as

\[ x(t) = \int_0^t H(t - \tau, \tau)f(\tau)d\tau \]  

\[ \text{(10)} \]

where \( H(t - \tau, \tau) \) denotes the impulse response function of the linear time-variant system.

The cross-correlation matrix of two arbitrary responses is expressed as

\[ R_{x_kx_l}(t) = E[x_k(t_k)x_l(t_l)] \]  

\[ \text{(11)} \]

Substituting equations (9) and (10) into equation (11) gives

\[ R_{x_kx_l}(t) = \int_0^{t_k} \int_0^{t_l} H(t_k - t_\tau, t_\tau)\Gamma(t_\tau)E[r(t_\tau)r^T(t_\tau)]\Gamma^T(t_\tau)H^T(t_l - t_\tau, t_\tau)d\tau_\tau d\tau_\tau \]  

\[ \text{(12)} \]

Using the Wiener–Khintchine theorem gives

\[ E[r(t_\tau)r^T(t_\tau)] = \int_{-\infty}^{+\infty} v^T S_f(\omega)e^{j\omega t_\tau}d\omega \]  

\[ \text{(13)} \]

where \( v = (e^{-j\omega t_1}, e^{-j\omega t_2}, \ldots, e^{-j\omega t_n})^T \), \( S_f(\omega) \) is the PSD of the random excitation and the superscript \( * \) denotes complex conjugate.

Substituting equation (13) into equation (12) gives

\[ R_{x_kx_l}(t) = \int_{-\infty}^{+\infty} I_k^T(\omega, t_k)S_f(\omega)I_l^T(\omega, t_l)d\omega \]  

\[ \text{(14)} \]

where

\[ I_k(\omega, t_k) = \int_0^{t_k} H(t_k - t_\tau, t_\tau)\Gamma(t_\tau)e^{j\omega t_\tau}d\tau_\tau, \]  

\[ I_l(\omega, t_l) = \int_0^{t_l} H(t_l - t_\tau, t_\tau)\Gamma(t_\tau)e^{j\omega t_\tau}d\tau_\tau. \]

Using the Wiener–Khintchine theorem, the cross-PSD of \( x_k(t) \) and \( x_l(t) \) is

\[ S_{x_kx_l}(\omega, t) = I_k^T(\omega, t_k)S_f(\omega)I_l^T(\omega, t_l) \]  

\[ \text{(15)} \]

By letting \( k = l \) in equation (15), the auto-PSD of the arbitrary response \( x_k \) is

\[ S_{x_kx_k}(\omega, t) = S_f(\omega) \int_0^{t_k} \int_0^{t_k} H(t - \tau_1, \tau_1) \times \Gamma(\tau_1)v^Tv^T(\tau_2)H^T(t - \tau_2, \tau_2)d\tau_1 d\tau_2 \]  

\[ \text{(16)} \]

Substituting the pseudo excitation17

\[ \tilde{f}(t) = \Gamma(t)r\sqrt{S_f(\omega)}e^{j\omega t} \]  

\[ \text{(17)} \]

into equation (10) gives the pseudo response

\[ \tilde{x}(\omega, t) = \int_0^{t} H(t - \tau, \tau)\Gamma(t)r\sqrt{S_f(\omega)}e^{j\omega \tau}d\tau \]  

\[ \text{(18)} \]
The auto-PSD of the response is calculated as
\[
S_{xx}(\omega, t) = \mathbf{x}^*(\omega, t)\mathbf{x}(\omega, t) = S_f(\omega) \int_0^T \mathbf{H}(t - \tau_1, \tau_1) \Gamma(\tau_1) \mathbf{r}^* \mathbf{r}^T(t_2) \times \mathbf{H}^T(t - \tau_2, \tau_2) d\tau_1 d\tau_2
\]  
(19)
i.e. the same as equation (16). For a linear time-variant system, the PSD of an arbitrary non-stationary random response can be calculated by PEM.

**PEM applied to the coupled vehicle–slab track system**

The coupled vehicle–slab track system is excited by multipoint random external forces from the same track irregularity. A time lag exists between two arbitrary random external forces. The track irregularity is assumed to be a Gaussian random process with zero mean value. \(^6\) The random excitation of the track irregularity is expressed as
\[
r(t) = [r(t - t_1), r(t - t_2), r(t - t_3), r(t - t_4)]^T
\]  
(20)
where \(t_1 = 0, t_2 = 2l_v/v, t_3 = 2l_v/v\) and \(t_4 = 2(l_1 + l_v)/v, l_v\) denotes half the distance between the two bogie centres; \(l_1\) denotes half the distance between the two axles of the same bogie and \(v\) is the vehicle speed.

The pseudo excitation of the track irregularity is introduced as
\[
\mathbf{\tilde{r}}(t) = \Gamma(t)(e^{-i\omega t_1}, e^{-i\omega t_2}, e^{-i\omega t_3}, e^{-i\omega t_4})^T \sqrt{S_f(\omega)} e^{i\omega t}
\]  
(21)
where \(S_f(\omega)\) denotes the PSD of the track irregularity and \(\omega\) is the circular frequency.

According to equations (10) and (21), an arbitrary non-stationary pseudo response \(\mathbf{\tilde{u}}(\omega, t)\) is obtained, and then the corresponding PSD is calculated as
\[
S_{uu}(\omega, t) = \mathbf{\tilde{u}}^*(\omega, t)\mathbf{\tilde{u}}(\omega, t)
\]  
(22)

For practical problems, the PSD in equation (22) is calculated over a specified frequency range. The standard deviation (SD) of the arbitrary response is calculated as
\[
\sigma_u = \sqrt{\int_{\omega_1}^{\omega_2} S_{uu}(\omega, t) d\omega}
\]  
(23)

where \(\omega_1\) and \(\omega_2\) are the minimum and maximum computational frequencies, respectively.

**Parallel algorithm for PEM**

The specified frequency range is divided into a sequence of equally spaced frequency points, at each of which the PSD in equation (22) is calculated. For a large-scale coupled vehicle–slab track system, the computational cost of the traditional PEM is high and increases with the number of frequency points. However, the random responses of the vehicle and slab track are calculated independently at each frequency point, with no exchange of data between different frequency points. Thus, PEM is well suited for parallel computation. Generally, the number of processors is less than the number of frequency points, so that a parallel algorithm is easily derived by letting each processor independently calculate the random responses of the vehicle and slab track at a specified number of frequency points. This parallel computation process is shown in Figure 2. First, the frequency points are distributed as equally as possible to each processor. Second, the random responses of the vehicle and slab track are calculated by each processor at its specified frequency points. Finally, the results calculated by each processor are merged to give results at all frequency points.

The speed-up ratio is defined as the ratio of CPU time of the serial computation to that of the parallel computation. The theoretical speed-up ratio is equal to the number of processors used. However, it is difficult to obtain this theoretical value due to the additional cost of set-up and overhead calculations and because some processors may have one more

![Flowchart for parallel computation.](image-url)
frequency point than others. One node with up to eight processors was used to investigate the speed-up ratios for different numbers of processors, for the coupled vehicle–slab track system for which the parameters are given in the section ‘Numerical results and discussion’. The number of frequency points was 40. Accurate speed-up ratios were obtained for different numbers of processors by using the Windows operating system on a 2.67 GHz CPU HP Z800 which had one node with eight processors. The results are given in Table 1, from which it is seen that the ratios increase almost linearly with the number of processors. If the number of processors exceeds the number of computational frequency points, a more complex parallel algorithm would need to be developed for the step-by-step integration at each frequency point.

**Numerical results and discussion**

A 1.86 GHz CPU Lenovo Shenteng 7000 with eight nodes each having eight processors was used to compute the random response of the coupled vehicle–slab track system. The vehicle starts at the left-hand end of a 100 m long slab track, with a speed of 200 km/h (except in the section ‘Effect of vehicle speed on the random response of the coupled vehicle–slab track system’) and the steady state response of the coupled vehicle–slab track system were obtained at the middle of the slab track thereby ensuring that the end conditions did not influence the obtained results. The vehicle and slab track parameters are listed in Tables 2 and 3. The random vertical track irregularity PSD was adopted to give

\[
S(\Omega) = \frac{A_v \Omega_r^2}{(\Omega^2 + \Omega_r^2)(\Omega^2 + \Omega_c^2)} \text{m}^2/(\text{rad/m})
\]

where \(\Omega_c\) and \(\Omega_r\) are cutoff frequencies, \(A_v\) is the roughness constant and \(\Omega\) is the spatial circular frequency. The values used were \(\Omega_c = 0.8246\ \text{rad/m}, \text{ and } \Omega_r = 0.0206\ \text{rad/m}\) and \(A_v = 4.032 \times 10^{-7}\ \text{m-rad}\), with a computational frequency range of between 0.5 and 100 Hz and 200 frequency points. Five alternative sets of rail pad, CAM and FSL parameters were used, see Table 4.

**Verification by Monte Carlo method**

The random response of the coupled vehicle–slab track system was calculated both by PEM and by the Monte Carlo method with 10, 100 and 500 spatial samples. The results are compared in Figure 3 which gives the SDs of the wheel–rail force, bogie vertical acceleration and rail acceleration, from which it is seen that the Monte Carlo results become close to those calculated by PEM as the number of samples is increased. For 500 spatial samples the maximum SD differences are 6.8, 8.5 and 9.5%, for the wheel–rail force, bogie vertical acceleration and rail acceleration, respectively. Note that the wheel–rail and bogie results are periodic due to the slab track stiffness changing periodically, with periodicity equal to the slab length, i.e. 5 m.

**Random response of the coupled vehicle–slab track system**

The wheel–rail force plays a key role for the coupled vehicle–slab track system and so Figure 4 shows the

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**Table 2. Vehicle parameters.**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car body mass</td>
<td>39,600 kg</td>
</tr>
<tr>
<td>Car body pitch moment of inertia</td>
<td>1,940,400 kg·m²</td>
</tr>
<tr>
<td>Bogie mass</td>
<td>3200 kg</td>
</tr>
<tr>
<td>Bogie pitch moment of inertia</td>
<td>1752 kg·m²</td>
</tr>
<tr>
<td>Wheelset mass</td>
<td>2000 kg</td>
</tr>
<tr>
<td>Primary suspension stiffness</td>
<td>2352 kN/m</td>
</tr>
<tr>
<td>Primary suspension damping</td>
<td>39.2 kN·s/m</td>
</tr>
<tr>
<td>Secondary suspension stiffness</td>
<td>411 kN/m</td>
</tr>
<tr>
<td>Secondary suspension damping</td>
<td>19.6 kN·s/m</td>
</tr>
<tr>
<td>Distance between two bogie centres</td>
<td>17.5 m</td>
</tr>
<tr>
<td>Distance between two axles of same bogie</td>
<td>2.5 m</td>
</tr>
<tr>
<td>Wheel rolling radius</td>
<td>0.43 m</td>
</tr>
</tbody>
</table>

**Table 3. Slab track parameters.**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rail mass per unit length</td>
<td>60.64 kg/m</td>
</tr>
<tr>
<td>Area moment of the rail cross-section</td>
<td>3.217 x 10⁻⁵ m⁴</td>
</tr>
<tr>
<td>Rail elastic modulus</td>
<td>2.059 x 10¹¹ N/m²</td>
</tr>
<tr>
<td>Rail pad stiffness</td>
<td>6.0 x 10⁷ N/m</td>
</tr>
<tr>
<td>Rail pad damping</td>
<td>3.625 x 10⁴ N·s/m</td>
</tr>
<tr>
<td>Slab length</td>
<td>5 m</td>
</tr>
<tr>
<td>Slab width</td>
<td>2.4 m</td>
</tr>
<tr>
<td>Slab height</td>
<td>0.19 m</td>
</tr>
<tr>
<td>Slab elastic modulus</td>
<td>42,000 MPa</td>
</tr>
<tr>
<td>CAM length</td>
<td>5 m</td>
</tr>
<tr>
<td>CAM width</td>
<td>2.4 m</td>
</tr>
<tr>
<td>CAM height</td>
<td>0.05 m</td>
</tr>
<tr>
<td>CAM elastic modulus</td>
<td>100 MPa</td>
</tr>
<tr>
<td>CAM damping¹²</td>
<td>8.3 x 10⁴ N·s/m</td>
</tr>
<tr>
<td>Roadbed length</td>
<td>5 m</td>
</tr>
<tr>
<td>Roadbed width</td>
<td>2.8 m</td>
</tr>
<tr>
<td>Roadbed height</td>
<td>0.3 m</td>
</tr>
<tr>
<td>Roadbed elastic modulus</td>
<td>42,000 MPa</td>
</tr>
<tr>
<td>FSL length</td>
<td>100 m</td>
</tr>
<tr>
<td>FSL width</td>
<td>6.2 m</td>
</tr>
<tr>
<td>FSL height</td>
<td>0.4 m</td>
</tr>
<tr>
<td>FSL elastic modulus</td>
<td>220 MPa</td>
</tr>
<tr>
<td>FSL damping</td>
<td>8.3 x 10⁴ N·s/m</td>
</tr>
</tbody>
</table>

---
wheel–rail force PSDs for the first two wheelsets, which also vary periodically with time due to the periodic variation of the slab track stiffness. Within each such cycle, the wheel–rail force PSD varies quasi-periodically due to the rail being supported discretely by the rail pads, so that its quasi-periodic length is equal to the rail pad spacing, i.e. 0.625 m (a similar phenomenon occurs in Figure 5(b) and (c)). The main peak PSD values for the first two wheel–rail forces differ because of bogie pitch and occur at approximately 50.5 and 51 Hz, respectively, with their maximum fluctuations being $0.79 \times 10^6$ and $0.27 \times 10^6$ N$^2$/Hz, respectively. The wheel–rail force PSDs for the third and fourth wheelsets are similar to those for the first two wheelsets, respectively. The wheel–rail force PSD distribution is mainly affected by the vehicle speed and the rail pad stiffness, see the next two subsections.

Table 4. Alternative rail pad, CAM and FSL parameters.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Case</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rail pad Stiffness (10$^6$ N/m)</td>
<td></td>
<td>20</td>
<td>40</td>
<td>60</td>
<td>80</td>
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Figure 3. Comparison between PEM and Monte Carlo method: (a) wheel–rail force SD; (b) bogie vertical acceleration SD; and (c) rail acceleration SD.
Figure 5(a) to (c) show the acceleration PSDs of the vehicle’s components. Their maximum amplitudes decrease dramatically from wheelset to car body due to the effectiveness of the primary and secondary suspensions, particular the former, so that the vertical acceleration PSD of the car body varies only slightly with time. Thus, the vertical acceleration PSD of the car body is hardly affected by the slab track stiffness. Its maximum value occurs at approximately 0.99 Hz, see Figure 5(a), which is close to its vertical natural frequency. Figure 5(b) shows that the four peak values for the first bogie occur at approximately 6, 20, 47 and 61.5 Hz, respectively, the first of which is close to the vertical natural frequency of the bogie. The three other peak values are mainly related to vehicle speed, see the next subsection. Compared with the first two peak values, the third and fourth peak values clearly vary more with time. The second bogie vertical acceleration PSD is similar to that of the first bogie and so is not shown. The first wheelset acceleration PSD is shown in Figure 5(c) and the wheel–rail force PSD for the first wheelset behaves similarly.

Figure 4. Wheel–rail force PSD for (a) the first wheelset and (b) the second wheelset.
Figure 5(d) to (f) show the PSDs of the accelerations of the slab track components which were obtained at 61.875 m from the left-hand end of the slab track. Their maximum amplitudes decrease from rail to roadbed with the rail pad clearly playing an important role in isolating the vibration. From Figure 5(d) the main frequency of the rail acceleration PSD is seen to occur at approximately 51.5 Hz (which approximates the frequency of the maximum PSD of the wheel–rail force), and its two main peak values are induced by the first two wheelsets passing over the rail, with the distance between them being equal to the distance between the axles of the bogie, i.e. 2.5 m. The peak value induced by the first wheel–rail force exceeds that for the second wheel–rail force because the first wheel–rail force exceeds the second one. There are also small peak values between 70 and 100 Hz. The PSD of the slab and roadbed accelerations are respectively shown in Figure 5(e) and (f) and their maximum peak values both occur at 89.5 Hz. This frequency is not induced by the frequency of the maximum PSD of the wheel—rail force, which is isolated by the rail pad.
Effect of vehicle speed on the random response of the coupled vehicle–slab track system

The vehicle speed was varied between 200 and 350 km/h at intervals of 50 km/h to investigate the effect of vehicle speed on the random response of the coupled vehicle–slab track system. Figure 6(a) shows variation with vehicle speed of the maximum PSD of the first two wheel–rail forces and the numbers on it are the frequency of each maximum; these numbers vary only to a small extent except for the first wheel–rail force at 250 km/h. The variation of the maximum PSD of the first wheel–rail force is clearly the more severe of the two between 300 and 350 km/h. The variation of the wheelset acceleration PSD is not shown because it is similar to that of the wheel–rail force PSD.

In Figure 6(b) the regions A and B represent the variation of the first two peak values of the car body’s vertical acceleration PSD with vehicle speed. In region A, the peak values increase with vehicle speed, although their frequencies are almost identical for the first wheel–rail force at 250 km/h. The variation of the maximum PSD of the first wheel–rail force is clearly the more severe of the two between 300 and 350 km/h. The variation of the wheelset acceleration PSD is not shown because it is similar to that of the wheel–rail force PSD.

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Figure 6(c) shows the variation of the vehicle speed of the maximum PSD of the accelerations of the slab track components. The peak value for the rail increases with vehicle speed, but the values for the slab and roadbed at 300 km/h are lower than at 350 km/h. The considered maximum frequency of the excitation is only 100 Hz due to lack of knowledge towards the right as the vehicle speed increases, with the peak values being altered very little, since they are caused by the phase difference of the excitation of the track irregularity, which depends on the vehicle speed.

In Figure 6(c) the peak values of the bogie vertical acceleration PSDs at 200, 250, 300 and 350 km/h are denoted by A, B, C and D, respectively, and their values increase as the vehicle speed increases. The frequencies of the first peak values are close to the vertical natural frequency of the bogie and so are little influenced by the vehicle speed and they become larger with increasing vehicle speed, so that at higher speeds the frequency of the maximum is mainly precipitated by the natural frequency of the bogie. The frequencies and amplitudes of other peak values also change with vehicle speed, and are again caused by the phase difference of the excitation of the track irregularity. Such complex phenomena were also reported by Zhang et al.16

Figure 6(d) shows the variation of the vehicle speed of the maximum PSD of the accelerations of the slab track components. The peak value for the rail increases with vehicle speed, but the values for the slab and roadbed at 300 km/h are lower than at 350 km/h. The considered maximum frequency of the excitation is only 100 Hz due to lack of knowledge towards the right as the vehicle speed increases, with the peak values being altered very little, since they are caused by the phase difference of the excitation of the track irregularity, which depends on the vehicle speed.
about the track irregularity PSD for a high-speed line at high frequency, which may excite the higher frequencies of the PSD of the slab and roadbed accelerations at 350 km/h.

**Effect of slab track parameters on the random response of the coupled vehicle–slab track system**

Figure 7(a) shows that the maximum PSD of the first two wheel–rail forces increase with rail pad stiffness whereas Figure 7(b) shows that they decrease with rail pad damping. The maximum PSDs of the first two wheel–rail forces increase by 469.1 and 359.2%, respectively, when the rail pad stiffness is increased five-fold. The numbers printed on Figure 7(a) are the frequencies of the maximum PSD and can be seen to increase with rail pad stiffness, whereas Figure 7(b) shows that the rail pad damping hardly alters the frequency of the maximum.

The variation range of the maximum PSD of the car body vertical acceleration was between 0.0245 and 0.0246 m^4/s^2/Hz as rail pad stiffness increased and so this maximum is unaffected by the rail pad parameters.

Figure 7(c) shows that the variance of the first peak value of the bogie vertical acceleration PSD is...
relatively small due to it being mainly induced by the natural frequency of the bogie, while the variance of other peak values is complex. Figure 7(d) shows that only the third peak decreases as the rail pad damping increases.

Figure 7(e) shows that the maximum PSD of the rail, slab and roadbed accelerations increase with the pad stiffness. Figure 7(f) shows that the frequencies of the maximum PSD of the rail and roadbed accelerations are not affected significantly by increasing the rail pad damping and nor is the frequency of the maximum PSD of the slab acceleration, except for the first value. The rail pad damping affects the maximum rail acceleration PSD much more than the slab and roadbed acceleration PSDs.

Figure 8 shows the effects of CAM and FSL stiffness and damping on vehicle and slab track components, showing whichever is appropriate of acceleration or wheel–rail force PSD. It can be seen that the FSL stiffness has much the greatest effect except for the rail, although the FSL damping can also be significant. Frequency values are not printed on Figure 8; however, the frequency of the maximum PSD of the bogie vertical acceleration changed from...
47 to 60 Hz when the CAM stiffness changed from 200 to 400 MPa. In contrast, the frequencies of the maximum PSD of the vehicle and slab track component accelerations are hardly affected by the other CAM and FSL parameters.

Conclusions

A coupled vehicle–slab track model has been developed to investigate the non-stationary random vibration of coupled vehicle–slab track systems. The vehicle is treated as a multibody model, in which the elasticity of the car body, bogie and wheelset are neglected. The slab track is flexible with the rail, slab and roadbed modeled by Bernoulli–Euler beam theory. The vehicle and slab track are coupled through the linearized wheel–rail force, and the non-stationary random vibration of the coupled vehicle–slab track system is calculated by combining PEM with step-by-step integration. A parallel algorithm based on PEM was adopted to improve computational efficiency. The paper investigated how the random responses of the vehicle and slab track are influenced by the vehicle speed and the slab track parameters, leading to the following conclusions.

1. The parallel algorithm based on PEM is easily implemented and compared with serial computation; it significantly improves computational efficiency when calculating the random response of the coupled vehicle–slab track system.

2. The random response of the car body’s vertical acceleration is time-invariant with the maximum vertical acceleration PSD of the car body being induced by its vertical natural frequency. The PSD of the vertical accelerations of the bogie and wheelset vary periodically with time as also does the wheel–rail force. The periodic lengths of these PSD are equal to the slab length. The main frequency of the rail acceleration PSD is close to the frequency of the maximum PSD of the wheel–rail force.

3. The PSD of the vertical accelerations of the car body, bogie and rail increase with vehicle speed, as also does the first wheel–rail force. The vehicle speed barely affects the frequency of the maximum vertical acceleration PSD of the car body, wheelset and rail.

4. Compared with the CAM and FSL stiffness and damping, the stiffness and damping of the rail pad has much more influence on the random response of the coupled vehicle–slab track system. The damping of the rail pad evidently reduces the maximum vertical acceleration PSD of the bogie, wheelset and rail. However, the corresponding frequencies of these maximum PSDs hardly vary. For the case of the CAM and FSL parameters, the maximum vertical acceleration PSDs of the bogie, slab and roadbed are evidently most affected by the FSL stiffness compared with the other parameters.

5. Study of track irregularity for high-speed lines at high frequency needs to be developed further to better understand the random responses.

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Conflict of interest

There are no conflicting interests associated with the financial support of this paper.

References


Appendix I

Notation

- $A_v$: roughness constant
- $C$: damping matrix of the linear time-variant system
- $C_t, C_r, C_{vt}$: damping matrices of the slab track, vehicle and coupled vehicle–slab track models
- $E[ ]$: expectation operator
- $f$: random external force vector
- $f_r, f_r, f_{vt}$: external load vectors of the slab track, vehicle and coupled vehicle–slab track models
- $G$: Hertzian constant
- $H$: impulse response function of the linear time-variant system
- $k_h$: contact stiffness of the wheel and rail
- $K$: stiffness matrix of the linear time-variant system
- $K_t, K_r, K_{vt}$: stiffness matrices of the slab track, vehicle and coupled vehicle–slab track models.
- $l_c, l_t$: half the distances between two bogie centres and axles of the same bogie
- $M$: mass matrix of the linear time-variant system
- $M_t, M_r, M_{vt}$: mass matrices of the slab track, vehicle and coupled vehicle–slab track models.
- $n, N$: numbers of excitations and frequency points
- $P_0$: static wheel load
- $r$: random excitation
- $R$: wheel radius
- $S_f$: PSD of the random excitation
- $S_r$: PSD of the track irregularity
- $t$: time lag
- $v$: vehicle speed
- $x$: track irregularity location
- $x_t, x_r, x_w$: rail displacement at wheel position and wheel displacement
- $x_t, x_r, x_{vt}$ and $x_t, x_r, x_{vt}$: displacement, velocity and acceleration vectors of the slab track, vehicle
- $x_t, x_r, x_{vt}$: coupled vehicle–slab track models
- $!$: index matrix
- $\phi$: random phase angles
- $\omega$: circular frequency
- $\omega_1, \omega_2$: minimum and maximum computational frequencies
- $\omega_k$: circular frequency within the computational frequency range
- $\Delta \omega$: circular frequency interval
- $\Omega$: spatial circular frequency
- $\Omega_c, \Omega_r$: cutoff frequencies